Helical shell models for three dimensional turbulence.

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February 5, 2008

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Abstract

In this paper we study a new class of shell models, defined in terms of two complex dynamical variables per shell, transporting positive and negative helicity respectively.

The dynamical equations are derived from a decomposition into helical modes of the velocity Fourier components of Navier-Stokes equations (F. Waleffe, Phys. Fluids A **4**, 350 (1992)).

This decomposition leads to four different types of shell models, according to the possible non-equivalent combinations of helicities of the three interacting modes in each triad.

Free parameters are fixed by imposing the conservation of energy and of a "generalized helicity" H_α in the inviscid and unforced limit.

For $\alpha = 1$ this non-positive invariant looks exactly like helicity in the Fourier-helical decomposition of the Navier-Stokes equations.

Long numerical integrations are performed, allowing the computation of the scaling exponents of the velocity increments and energy flux moments.

The dependence of the models on the generalized helicity parameter α and on the scale parameter λ is also studied.

PDEs are finally derived in the limit when the ratio between shells goes to one.

1 Introduction.

The most intriguing problem in 3 dimensional turbulence is related to the understanding of the dynamical mechanism triggering and supporting the energy cascade from large to small scales. According to the celebrated Kolmogorov theory [1] (K41), energy should be transferred downwards in scales following a self-similar and homogeneous process. This idea, plus the assumption of local isotropy and small scales universality, led Kolmogorov to formulate a precise prediction on the scaling properties of the increments of turbulent velocity fields, $\delta v(l) \sim |v(x+l) - v(x)|$, at distances l. Namely, following Kolmogorov and denoting by $\epsilon(l)$ the energy transfer through scale l, we have:

$$S_p(l) \equiv \langle (\delta v(l))^p \rangle = C_p \langle (\epsilon(l))^{p/3} \rangle l^{p/3} \sim l^{\zeta(p)} \tag{1}$$

where C_p are constants and the scale l is supposed to be in the inertial range, i.e. much smaller than the integral scale and much larger than the viscous dissipation cutoff. From (1) we deduce that depending on the scaling properties of the energy transfer, we can observe different power law behaviors for the structure functions $S_p(l)$. Kolmogorov made the simplest assumption by taking $\epsilon(l)$ to be constant and deducing that:

$$\zeta(p) = p/3, \forall p \tag{2}$$

While from a qualitative point of view the Kolmogorov intuition was a true breakthrough in the understanding of turbulence, his theory lacks quantitative agreement with experiments. In particular, there are many experimental and numerical [2, 3, 4] results telling us that energy is transferred intermittently, leading to non-trivial scaling corrections to the "p over 3" Kolmogorov prediction for the $\zeta(p)$ exponents.
More and more efforts have been devoted in trying to understand and to model with good accuracy the energy transfer dynamics.
Beside analytical and direct numerical approach, there are two other possible choices: either building simple random processes for the chaotic energy transfer among different scales [5, 6, 7, 8] or studying a dynamical deterministic model. In the last 20 years, among the possible dynamical modelizations of fully developed energy transfer, those obtained by using *shell models* have been particularly successfully. These models concentrate only on the dynamical interactions among degrees of freedom at different scales, neglecting completely their spatial location. In this way a one-dimensional chain of interacting Fourier modes is constructed. The same non-linear structure of Navier-Stokes eqs. is retained, but their three-dimensional vectorial nature is completely lost. The simplifi-

cations are so strong that only *a posteriori* one can say whether the model is interesting and reliable or not.

For the most famous among all shell models, the Gledzer-Ohkitani-Yamada (GOY) model, are now available a wide number of works and contributes ([9]-[15]). The intermittent and chaotic dynamics of the GOY model is remarkably close (for a suitable choice of the free parameters) to what is found for the Navier-Stokes eqs.

The most striking property of the GOY model is that, for some particular choices of the free parameters, it has an intermittent and chaotic dynamics quantitatively close to what is found for the Navier-Stokes equations. In particular, the set of scaling exponents $\zeta(p)$ coincides (for a suitable choice of parameters) with that measured in true turbulent flows.

Recently, it has been pointed out that the GOY capability of having a chaotic and intermittent behavior with scaling exponents very close to the real ones, is strictly related to the conservation in the inviscid, unforced limit of two quadratic quantities. The first quantity is the *energy*, while the second is, roughly speaking, the equivalent of *helicity* in 3D turbulence [14].

Both the *GOY-helicity* [15] and the helicity in the N-S equations [16, 17] have been suggested to play an important role in triggering the intermittent nature of the energy cascade.

This new emphasis on the role played by the helicity motivated us to study a new class of shell models. These models can be seen as a generalization of the GOY model such that their helical structure is closer to the corresponding N-S case (indeed, one of them is nothing but the old GOY case).

We study shell models with two complex variables per shell, derived on the basis of an helical formulation of the N-S equations in the Fourier space.

In this way, it is possible to obtain a second non-positive defined invariant closer to the definition of helicity in the N-S equations. Our aim consists in trying to understand the importance of both (competing?) transfer of energy and helicity in N-S eqs. by examining the non trivial dynamics shown by this new class of helical-shell models.

The outline of the paper is the following: in section 2 we review the GOY model; in section 3 we introduce the new class of helical-shell models; in section 4 the basic triad interactions within three contiguous shells are studied. Section 5 contains the results of our numerical simulations; in section 6 PDEs for the continuum limit (ratio between shells that goes to one) is derived; conclusions follow in section 7.

2 The GOY model.

The GOY model can be seen as a severe truncation of the Navier-Stokes equations: it retains only one complex mode u_n as a representative of all Fourier modes in the shell of wave numbers k between $k_n = k_0 \lambda^n$ and k_{n+1}, λ being an arbitrary scale parameter ($\lambda > 1$), usually taken equal to 2.
The dynamics is governed by the following set of complex coupled ODEs where only couplings with the nearest and next nearest shells are kept:

$$\frac{d}{dt} u_n = i\, k_n \left(a u_{n+1}^* u_{n+2}^* + b u_{n+1}^* u_{n-1}^* + c u_{n-1}^* u_{n-2}^* \right) - \nu k_n^2 u_n + \delta_{n,n_0} f, \quad (3)$$

where ν is the viscosity, f the external forcing acting on a large scale n_0 and a,b,c are three free parameters. By adjusting the time scale we can always fix $a = 1$; the possible choices for b,c are restricted by imposing the conservation of two quadratic quantities in the inviscid and unforced limit:

$$W_{1,2} = \sum_n z_{1,2}^n |u_n|^2, \quad (4)$$

where $z_{1,2}$ are the solutions of the quadratic equation:

$$c\lambda^2 z^2 + b\lambda z + a = 0. \quad (5)$$

In order to stay as close as possible to the N-S equations, we require that one of the two conserved quantities is the energy, i.e. $z_1 = 1$:

$$W_1 = E = \sum_n |u_n|^2. \quad (6)$$

If we rewrite:

$$\begin{aligned} \lambda b &= -\epsilon, \\ \lambda^2 c &= \epsilon - 1, \end{aligned} \quad (7)$$

we are left with only two (λ and ϵ) of the original four free parameters. The second quadratic invariant is:

$$W_2 = H = \sum_n (\epsilon - 1)^{-n} |u_n|^2. \quad (8)$$

The characteristics of this second invariant change by changing ϵ: when $\epsilon < 1$ it is not positive-defined (as helicity in 3D), while if $\epsilon > 1$ it is positive defined (as enstrophy in 2D). Expression (8) can be rewritten as:

$$H_\alpha = \sum_n \chi(\epsilon)^n k_n^{\alpha(\epsilon,\lambda)} |u_n|^2, \tag{9}$$

where $\chi(\epsilon) = sign(\epsilon - 1)$ and the α parameter is related to ϵ and λ by:

$$|\epsilon - 1| = \lambda^{-\alpha}. \tag{10}$$

Our interest here is to consider how well the dynamics of a 3D turbulent flow is reproduced by the model: only ϵ in the range $0 < \epsilon < 1$ will be taken, in order to have a non-positive defined second invariant. Indeed, for $\alpha = 1$ our "generalized helicity" $H_\alpha = \sum_n (-1)^n k_n^\alpha |u_n|^2$ has physical dimensions coinciding with the 3D Navier-Stokes helicity. The two free parameters of the model can be taken to be λ (the ratio between adjacent shells) and α. The two coefficients b and c can be rewritten as:

$$\begin{aligned} b &= \lambda^{-\alpha-1} - \lambda^{-1}, \\ c &= -\lambda^{-\alpha-2}. \end{aligned} \tag{11}$$

Such a class of models has a highly non trivial dynamical behavior. Intermittency of the energy transfer and multifractal nature of energy dissipation have been studied in [9, 10, 11, 12, 13].

It turns out that the values of the $\zeta(p)$s are not universal, depending on the choice of ϵ and λ [12, 14]. Nevertheless, Kadanoff *et al.* [14] verified that the scaling exponents are invariant along the curve in the (ϵ, λ) plane where both energy and helicity are conserved, i.e. the curve at $\alpha(\epsilon, \lambda) = 1$.

This suggested that the second invariant plays a crucial role in the model dynamics.

More recently Biferale and Kerr [15] have attributed to the helicity the role of triggering the intermittent cascade of the energy from large to small scales.

These considerations together with the observation that this "GOY-helicity" is only partially consistent with the N-S helicity (i.e. it presents an asymmetry between odd and even shells that does not have any counterpart in physical flows) persuaded us to study a modified shell model [15], with two complex variables in each shell, carrying helicity of opposite sign, in order to obtain a second invariant closer to the N-S helicity. In the following section we introduce this new class of shell models, whose nonlinear interactions are constructed on the basis of a helical decomposition of the N-S equations in the Fourier space.

3   The helical-shell models.

In order to introduce two helical variables per shell we refer to the velocity field in N-S equations, expanded in a Fourier series [18]. The velocity vector can be

represented in terms of its projection on an orthogonal basis formed by $\mathbf{k}$, $\mathbf{h}_+$ and $\mathbf{h}_-$. The two basis vectors $\mathbf{h}_+$ and $\mathbf{h}_-$ can be chosen to be the eigenmodes of the curl operator:

$$i\mathbf{k} \times \mathbf{h_s} = sk\mathbf{h_s}, \tag{12}$$

where $s = \pm 1$.

This corresponds to an expansion of the velocity vector into helical modes:

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{u}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) = \sum_{\mathbf{k}} [u^+(\mathbf{k})\mathbf{h}_+ + u^-(\mathbf{k})\mathbf{h}_-] \exp(i\mathbf{k} \cdot \mathbf{x}). \tag{13}$$

The real flow velocity corresponding to the plus (minus) mode rotates clockwise (counterclockwise) as one moves in the direction of $\mathbf{k}$, thereby forming a left-handed (right-handed) helix; the vorticity vector of such a flow is parallel (antiparallel) to the velocity and the helicity is maximum (minimum).

The kinetic energy and helicity are given by:

$$\begin{aligned} E = \sum_{\mathbf{k}} E(\mathbf{k}) = \sum_{\mathbf{k}} \tfrac{1}{2}\mathbf{u}(\mathbf{k}) \cdot \mathbf{u}^*(\mathbf{k}) = \sum_{\mathbf{k}}(|\mathbf{u}^+(\mathbf{k})|^2 + |\mathbf{u}^-(\mathbf{k})|^2), \\ H = \sum_{\mathbf{k}} H(\mathbf{k}) = \sum_{\mathbf{k}} \tfrac{1}{2}\mathbf{u}(\mathbf{k}) \cdot \omega^*(\mathbf{k}) = \sum_{\mathbf{k}} k(|\mathbf{u}^+(\mathbf{k})|^2 - |\mathbf{u}^-(\mathbf{k})|^2). \end{aligned} \tag{14}$$

Plugging eq.(13) into the N-S equations yields to the dynamical evolution for the complex amplitudes $u^{s_k}(\mathbf{k}, t)$ ($s_k = \pm 1$) [18]:

$$\frac{d}{dt}u^{s_k}(\mathbf{k}) + \nu k^2 u^{s_k}(\mathbf{k}) = \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} \sum_{s_p, s_q} g_{\mathbf{k},\mathbf{p},\mathbf{q}}(s_p p - s_q q)(u^{s_p}(\mathbf{p})u^{s_q}(\mathbf{q}))^*. \tag{15}$$

The geometric factor $g_{\mathbf{k},\mathbf{p},\mathbf{q}} = -\tfrac{1}{4}(\mathbf{h_{s_k}} \wedge \mathbf{h_{s_p}} \cdot \mathbf{h_{s_q}})^*$ can be developed and factorized:

$$g = r\frac{s_k k + s_p p + s_q q}{p}, \tag{16}$$

where r is a function of the triad shape only [18].

Eight different types of interaction between three modes $u^{s_k}(\mathbf{k})$, $u^{s_p}(\mathbf{p})$, $u^{s_q}(\mathbf{q})$ with $|\mathbf{k}| < |\mathbf{p}| < |\mathbf{q}|$ are allowed according to the value of the triplet (s_k, s_p, s_q) = $(\pm 1, \pm 1, \pm 1)$: among them, only four are independent, the coefficients of the interaction with reversed helicities $(-s_k, -s_p, -s_q)$ being identical to those with (s_k, s_p, s_q) [18]:

1. $(s_k, s_p, s_q) = (+, -, +)$ or $(-, +, -)$,

2. $(s_k, s_p, s_q) = (+, -, -)$ or $(-, +, +)$,

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3. $(s_k, s_p, s_q) = (+, +, -)$ or $(-, -, +)$,

4. $(s_k, s_p, s_q) = (+, +, +)$ or $(-, -, -)$.

The equations corresponding to the single interaction (s_k, s_p, s_q) have the form (omitting viscosity and forcing):

$$\dot{u}^{s_k} = r(s_p p - s_q q) \frac{s_k k + s_p p + s_q q}{p} (u^{s_p} u^{s_q})^*,$$
$$\dot{u}^{s_p} = r(s_q q - s_k k) \frac{s_k k + s_p p + s_q q}{p} (u^{s_q} u^{s_k})^*, \qquad (17)$$
$$\dot{u}^{s_q} = r(s_k k - s_p p) \frac{s_k k + s_p p + s_q q}{p} (u^{s_k} u^{s_p})^*.$$

Each interaction independently conserves both energy and helicity on a single triad.

The dynamical system (17) composed by a single triad can be considered as the basic brick of the semi-infinite chain leading to the transfer of energy in turbulent flow.

By studying its stability properties it is possible to understand how energy and helicity are transferred among different wave-vectors belonging to the same triad.

Following [18], we distinguish two different kinds of dynamics: for the cases corresponding to the choices 1 and 3 of the triad helicities, the unstable wave-vector is the smallest-one, while for the cases 2 and 4 the unstable wave-vector is the medium-one. This very simple analysis suggests that by linking together a series of triads we should have a forward energy transfer for cases 1 and 3 and both forward and backward (competing) energy transfers for cases 2 and 4.

In a turbulent flow the direction of energy transfer is dynamically controlled by the triple correlation $< u^{s_k}(\mathbf{k})u^{s_p}(\mathbf{p})u^{s_q}(\mathbf{q}) >$. It is reasonable to argue that the statistical properties of $< u^{s_k}(\mathbf{k})u^{s_p}(\mathbf{p})u^{s_q}(\mathbf{q}) >$ are such that the overall direction in energy transfer coincides with the simplified behavior inferred from the stability study of the single triad (*instability assumption* in [18]). For instance, it is easy to estimate, by using the *instability assumption*, what would be the net energy transfer in the above four cases if the energy spectrum had the Kolmogorov scaling, $E(k) = k^{-5/3}$ [18]:

- 1 and 3: direct energy cascade from large to small scales,

- 4: reverse energy cascade from small to large scales,

- 2: direct (reverse) energy cascade for local (nonlocal) triads.

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The helical decomposition of the N-S eqs. suggested us the opportunity of defining a different GOY-like shell model for each one of the above four classes. In each shell we will have two complex dynamical variables u_n^+ and u_n^-, transporting positive and negative helicity respectively:

$$\dot{u}_n^+ = ik_n(a_j u_{n+2}^{s_1} u_{n+1}^{s_2} + b_j u_{n+1}^{s_3} u_{n-1}^{s_4} + c_j u_{n-1}^{s_5} u_{n-2}^{s_6})^* - \nu k_n^2 u_n^+ + \delta_{n,n_0} f^+,$$
$$\dot{u}_n^- = ik_n(a_j u_{n+2}^{-s_1} u_{n+1}^{-s_2} + b_j u_{n+1}^{-s_3} u_{n-1}^{-s_4} + c_j u_{n-1}^{-s_5} u_{n-2}^{-s_6})^* - \nu k_n^2 u_n^- + \delta_{n,n_0} f^-,$$
(18)

where $j = 1, ..., 4$ labels the four different models and the helicity indeces in the non linear interactions are easily found for each of the four cases (see table(1)). The coefficients a_j, b_j, c_j are determined imposing, as usually, the energy conservation:

$$\frac{d}{dt}E = \frac{d}{dt}(\sum_n (|u_n^+|^2 + |u_n^-|^2)) = 0, \tag{19}$$

that leads to the same relation for all models:

$$a_j + b_j\lambda + c_j\lambda^2 = 0 \tag{20}$$

By imposing also the conservation of the generalized helicity:

$$\frac{d}{dt}H_\alpha = \frac{d}{dt}\sum_n k_n^\alpha(|u_n^+|^2 - |u_n^-|^2) = 0 \tag{21}$$

we obtain different relations for the four models:

1. $a_1 - \lambda^{\alpha+1}b_1 + \lambda^{2(\alpha+1)}c_1 = 0$,

2. $a_2 - \lambda^{\alpha+1}b_2 - \lambda^{2(\alpha+1)}c_2 = 0$,

3. $a_3 + \lambda^{\alpha+1}b_3 - \lambda^{2(\alpha+1)}c_3 = 0$,

4. $a_4 + \lambda^{\alpha+1}b_4 + \lambda^{2(\alpha+1)}c_4 = 0$.

Fixing $a_j = 1$ one then finds the expressions for the coefficients b_j and c_j in terms of the parameters λ and α (see table(2)).
Let us remark two important facts. First, model 1 is nothing but two masked and uncorrelated versions of the original GOY model, with dynamical variables $(u_1^+, u_2^-, u_3^+, ...)$ and $(u_1^-, u_2^+, u_3^-, ...)$ respectively; rewriting the coefficients b_1 and c_1 in terms of the usual parameters λ and ϵ one can easily recover the standard GOY model coefficients. Second, also model 4 is formed by two independent sets of variables $(u_1^+, u_2^+, u_3^+, ...)$ and $(u_1^-, u_2^-, u_3^-, ...)$, each of them conserving separately a positive-definite quantity similar to enstrophy in 2D.

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Thus, model 4 is equivalent to two uncorrelated GOY models for 2D turbulence [19, 20].

The fact that the model 1 is formed by two uncorrelated GOY models is clearly due to our choice of taking only first and second-neighbor interactions. Model 4, on the other hand, will always be the sum of two separated models for any choice of the interacting modes composing the triads.

In the following, we will refer to the properties of model 1 intending corresponding properties of the GOY model. Model 4 will be studied only for completeness.

4   One-triad systems.

Following the *instability assumption* [18] that connects the single triad dynamics with the global statistical behavior of a multi-triads flow, we repeat the analog stability study for the three shells, single triad system.

By isolating three shells of waves numbers k_1, k_2, k_3, we can inspect their dynamical properties as determined by their mutual interactions.

For the positive-helicity modes we have:

$$
\begin{aligned}
\dot{u}_1^+ &= ik_1(u_3^{s_1} u_2^{s_2})^*, \\
\dot{u}_2^+ &= ik_2 b_j(u_3^{s_3} u_1^{s_4})^*, \\
\dot{u}_3^+ &= ik_3 c_j(u_2^{s_5} u_1^{s_6})^*,
\end{aligned} \tag{22}
$$

where $j = 1, ..., 4$ for the four models. An analogous set of equations holds for the negative-helicity modes, changing the sign of the helicity index of all variables.

This system conserves both energy and helicity.

The corresponding equations for the energies are the following:

$$
\begin{aligned}
\dot{E}_1 &= A, \\
\dot{E}_2 &= b_j \lambda A, \\
\dot{E}_3 &= c_j \lambda^2 A,
\end{aligned} \tag{23}
$$

where $A = 2k_1 \Im[(u_3^{s_1} u_2^{s_2} u_1^+) + (u_3^{-s_1} u_2^{-s_2} u_1^-)]$.

As found in [18], we know that the unstable mode is:

- the smallest mode for interactions 1 and 3,

- the medium mode for interactions 2 and 4.

In order to have a deeper understanding of the energy transfer dynamics, we have performed several integrations of eqs.(22), using the parameters values $\lambda = 2$, $\alpha = 1$, $k_1 = 2^{-4}$ and different initial conditions.

This analysis, performed on all four models, gives the following results:

- model 1: mode 1 gives energy equally to mode 2 and mode 3,

- model 2: mode 2 gives more energy to mode 3 and less to mode 1,

- model 3: mode 1 gives more energy to mode 2 and less to mode 3,

- model 4: mode 2 gives more energy to mode 1 and less to mode 3.

These energy exchanges are summarized in fig.(1).

Behaviors 1 and 4 have already been noticed by Ditlevsen and Mogensen [21] for the 3D and 2D GOY model respectively.

It is also interesting to investigate how these properties are modified when varying the α parameter in the models. Considering that in eq.(23) the sum of the three right-hand sides must be zero and normalizing to one the energy rate on the unstable shell, one can evaluate how the energy sharing between the other shells is affected by changing α (see fig.(2)):

- In model 1 there is a clear dependence on α. As this parameter grows up, more and more energy is captured by mode 2. For $\alpha > 1$, the energy gained by mode 2 become greater than the energy gained by mode 3, leading to a more local energy transfer;

- model 3 is remarkably independent of α, as we shall see in the following. This fact has very important consequences for the intermittent dynamics of the complete shell model;

- model 2 has a trend analogous to that of model 1, but with more drastic consequences: at $\alpha \sim 1.27$ the mode that receives most of the energy from the unstable mode becomes the first instead of the third-one. This would suggest a change in the direction of the flux: from downward to upward;

- in model 4 the mode that receives most of the energy remains mode 1, for all values of α, with a consequent reverse energy flux in all cases (as it must be, being model 4 a couple of 2D GOY models).

What emerges from this analysis is that the behavior of models seems to depend on the choice of the free parameter α, sometimes with strong consequences (as the direction of the flux in model 2). The only remarkable exception is the very low dependence of model 3.

Concerning helicity, we can consider the following equations:

$$\begin{aligned}
\dot{H}_1(\alpha) &= B, \\
\dot{H}_2(\alpha) &= \eta_2 b_j \lambda^{\alpha+1} B, \\
\dot{H}_3(\alpha) &= \eta_3 c_j \lambda^{2(\alpha+1)} B,
\end{aligned} \qquad (24)$$

where $B = 2k_1^{\alpha+1}\Im[(u_3^{s_1}u_2^{s_2}u_1^+) - (u_3^{-s_1}u_2^{-s_2}u_1^-)]$ and η_2 and η_3 depend on the particular model considered (see table(3)).

By performing the same analysis done for the energy evolution, one can conclude that helicity is transferred in different ways, as depicted in fig.(3).

Being helicity a non-positive defined quantity, forward (backward) transfer of positive (negative) helicity is equivalent to backward (forward) transfer of negative (positive) helicity. In view of this trivial remark, arrows in fig.(3) have only a visual value: indicating how helicity (with its own sign) is redistributed among shells.

Let us notice that models 1 and 3 show a very different pattern in the helicity exchange among shells. This can be the explanation of the very different scaling properties shown by the two models when varying α (see next section).

For example, we could argue that the dramatic dependence of the energy exchange on the α parameter in models 1 and 2, together with the well defined direction of the helicity transfer, can somehow enhance the role played by the second invariant with respect to the other two models.

In the next section we will check these arguments by performing a numerical study of eq.(18).

5 Numerical analysis (model 3).

In this section we will concentrate on the study of the statistical properties of model 3 compared with the already known results for the GOY model (model 1).

Model 3, at difference from model 2, shows for any value of α a forward energy transfer.

We integrated eqs.(18) for model 3 using the standard parameters $\alpha = 1$, $\lambda = 2$, $k_0 = 2^{-4}$, $f^\pm = 5(1+i)10^{-3}$, $n_0 = 1$, $\nu = 10^{-7}$ and a total number of shells equal to $N = 22$ and $N = 26$. In the numerical integration we used a fourth-order Runge-Kutta method, with a time step varying between $dt = 10^{-5}$ (for the

simulations with 22 shells) and 10^{-6} (for the cases with 26 shells). Most of the results presented here are for the case $N = 22$, with a number of iterations of the order of hundred millions, which correspond roughly to several thousands of eddy turnover times at the integral scale. Stationarity is checked by monitoring the total energy evolution.

The quantities we have looked at are the structure functions:

$$S_p(n) = \langle |\tilde{u}_n|^p \rangle, \tag{25}$$

where $\tilde{u}_n = \sqrt{|u_n^+|^2 + |u_n^-|^2}$.

In fig.(4) we show $\log_2[S_p(n)]$ as functions of $\log_2(k_n)$ for $p = 1, ..., 8$. There is a well defined inertial range where the structure functions follow a power law:

$$S_p(n) \sim k_n^{-\zeta(p)}. \tag{26}$$

Let us notice that in this model, at variance from the GOY model, there are not the period three oscillations superposed on the power-law scaling. In this case a linear least-square fit allows to compute the scaling exponents, $\zeta(p)$, with an uncertainty smaller than in the GOY model.

Nevertheless, in order to have a better estimate of the $\zeta(p)$s one can study the moments of a particular third-order quantity, the mean energy flux through the nth shell:

$$
\begin{aligned}
\Pi(n) = & \ 2k_n \Im[\langle u_{n+2}^{s_1} u_{n+1}^{s_2} u_n^+ \rangle + b_j \langle u_{n+1}^{s_3} u_n^+ u_{n-1}^{s_4} \rangle + \frac{1}{\lambda} \langle u_{n+1}^{s_1} u_n^{s_2} u_{n-1}^+ \rangle + \\
& + \langle u_{n+2}^{-s_1} u_{n+1}^{-s_2} u_n^- \rangle + b_j \langle u_{n+1}^{-s_3} u_n^- u_{n-1}^{-s_4} \rangle + \frac{1}{\lambda} \langle u_{n+1}^{-s_1} u_n^{-s_2} u_{n-1}^- \rangle].
\end{aligned}
\tag{27}
$$

As pointed out by Pisarenko *et al.* [13], one can write for this quantity the equivalent of the Kolmogorov's four-fifth law, expressing the balance between energy input and energy dissipation in the system.

Considering the energy variation over the first n shells:

$$
\begin{aligned}
\frac{d}{dt}[\sum_{m=1}^{n} (\langle |u_m^+|^2 \rangle + \langle |u_m^-|^2 \rangle)] = & \ -2\nu \sum_{m=1}^{n} k_m^2 (\langle |u_m^+|^2 \rangle + \langle |u_m^-|^2 \rangle) + \\
& + \Pi(n) + 2\Re[\langle f^+ u_{n_0}^{+\ *} \rangle + \langle f^- u_{n_0}^{-\ *} \rangle], \quad (28)
\end{aligned}
$$

and assuming a statistical steady state, in the limit of vanishing viscosity we are left with an inertial range in which $\Pi(n)$ is constant:

$$\Pi(n) \sim const. \tag{29}$$

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In analyzing the scaling properties of all our results we have always used Extended Self Similarity (ESS). ESS consists in plotting one structure function versus another ($S_3(n)$, for example). ESS turned out to improve the precision with which scaling exponents can be measured in true turbulent flows [3, 4] and in shell models [12].

We have applied ESS analysis to two kinds of fit:

1. $\log_2[S_p(n)]$ vs $\log_2[S_3(n)]$,

2. $\log_2[\Sigma_p(n)]$ vs $\log_2[\Sigma_3(n)]$,

where

$$\Sigma_p(n) = \langle |\Pi(n)/k_n|^{p/3} \rangle. \tag{30}$$

In all our simulations we have found the two sets of exponents coinciding within the numerical and statistical errors.

Fig. (5) shows the $\zeta(p)$s of model 3, compared with those of the GOY model with the same parameters $\alpha = 1$ and $\lambda = 2$ (corresponding to the classical choice which conserves the analog of the 3D helicity). The scaling is nearly the same. Indeed, we argued in the previous section that both models have a forward energy flux; what turned out to be different was the exchange of helicity among shells, together with a different sensitivity on the parameter α connected to this second invariant.

What occurs is a strong similarity for $\alpha = 1$; nevertheless we expect a different behavior when this parameter is allowed to vary.

5.1   The α dependence

For the α dependence of the models we have explored two other different values: $\alpha = 0.5$ and $\alpha = 2$ (keeping fix $\lambda = 2$).

It is well known [12, 14, 15] that the GOY model shows a strong dependence of its statistical properties on the α value. For example, if $\alpha < 2/3$ the dynamics is attracted toward a fixed point with Kolmogorov scaling, $\zeta(p) = p/3$. For $\alpha > 1$ intermittency become more important than what is usually measured in turbulent flows [14].

On the other hand, the statistical properties of model 3 turn out to be robust under changes of the α parameter.

In fig.(6) we show the $\zeta_\alpha(p)$ exponents for the GOY model and model 3 at $\alpha = 0.5, 1, 2$. Clearly, there is an evident dependence of the $\zeta(p)$s on α for the GOY case while for the model 3 the different exponents coincide within numerical errors.

This behavior is in perfect agreement with the phenomenological speculations argued in the previous section from the study of the single-triad system.

The robustness of model 3, with respect to variation in α, gives to this model an important role among the possible shell models of turbulence.

5.2   The λ dependence

Concerning the dependence on the scale parameter λ, we have performed an exploratory study by fixing $\alpha = 1$ and taking $\lambda = 1.5$ and $\lambda = 2.5$ in model 3. For the case with $\lambda = 2.5$ the $\zeta(p)$ exponents are still stuck to the previous values at $\lambda = 2$. On the other hand, for the case $\lambda = 1.5$ we found a week discrepancy comparable with the one found in [14] for the GOY model. The issue of what happens in the limit $\lambda \to 1$ (the so-called continuum limit) is one of the most intriguing problems that must be analyzed in both cases, GOY model and model 3 (see next section).

In fig.(7) we show three sets of exponents obtained for different choices of λ for both model 3 and the GOY model.

The origin and significancy of the weak spreading in the values of $\zeta(p)$s is far from being understood. By changing λ, one changes the ratio between adjacent shells and therefore how viscous and inertial ranges match together. This non trivial matching maybe interferes also with the determination of the scaling exponents [22].

The λ dependence in all this shell models is however a very important open question due to the obvious interest in having a PDE describing the continuum limit ($\lambda \to 1$) of the energy transfer. In the following section we derive the equation for the continuum limit of all four models and we present some proposal for further investigations.

6   The continuum limit

As anticipated in the previous section one of the most interesting and still unexplored aspect of GOY-like shell models is their dynamics in the continuum limit [23].

For continuum limit we intend the limit when the separation between shells goes to one, i.e. $\lambda \to 1$:

$$k_{n+1} = \lambda k_n \sim (1 + \delta)k_n, \tag{31}$$

where we have defined $\lambda = \exp(\delta) \sim 1 + \delta + O(\delta^2)$. In the limit (31) we can expand the u_n set as follows:

$$u(k_{n+m}) = u(k_n) + \delta \, m \, k_n \partial_k u(k_n) + O(\delta^2). \qquad (32)$$

Taking into account the equivalent expansions for the a, b, c coefficients (see table 2) and after some simple algebra one realizes that all the three models (model 1,2 and 3) lead to the same expression in the continuum, namely:

$$
\begin{aligned}
\partial_t u^+(k) &= ik \left(4ku^- \partial_k u^+ + 2ku^+ \partial_k u^- + (2+\alpha)u^+ u^- - \alpha u^- u^- \right)^* + \\
&\quad -\nu k^2 u^+ + f(k),
\end{aligned} \qquad (33)
$$

the corresponding equation for the u^-s is obtainable from (33) by changing all helicity indeces.

This PDE worths a deeper study for many reasons.

First, let us notice that the continuum limit is highly non reversible, i. e. trying to come back to a logarithmically-equispaced shell structure one does not recover the original equations (18). Second, the continuum model shows an unexpected universality: it is the limit of three models which have very different behaviors at $\lambda > 1$. Third, even in the continuum there are two conserved quantities (in the unforced and inviscid limit) corresponding to the continuum analogous of energy and generalized helicity:

$$E = \int \frac{dk}{k}(|u^+|^2 + |u^-|^2), \quad H_\alpha = \int \frac{dk}{k} k^\alpha (|u^+|^2 - |u^-|^2), \qquad (34)$$

where the, apparently unusual, dk/k integration step comes from the original logarithmically- equispaced shell structure. Let us remark that the most interesting difference between the continuum expression (33) and the analog for the old GOY model is that now in (33) also helicity conservation is well defined. This was not the case for the continuum GOY model. The apparent paradox (model 1 is formed by two uncorrelated GOY models when $\lambda > 1$) is easily solved by noticing that in the continuum limit shells collapse in such a way that the original ordering is destructed. This limiting procedure introduces a coupling between the two sub-models.

This drastic difference with the GOY continuum case suggests the possibility that this new set of PDEs has a much richer dynamics than the corresponding GOY PDEs. In that case, indeed, is quite easy to realize that PDEs are integrable along the characteristics [24] (at least for the case of real variables). The solutions have a burst-like shape with a Kolmogorov scaling, reaching infinite k at finite time (for zero viscosity).

Whether eqs. (33) are more interesting or not is still an open question.

As for the continuum limit of model 4 we need to go the the second order in the δ expansion and we obtain:

$$\partial_t u^+(k) = ik(16k^2(\partial_k u^+)^2 + 2k^2 u^+ \partial_k^2 u^+ + 12(\alpha+2)ku^+ \partial_k u^+ +$$
$$+(\alpha+2)^2(u^+)^2)^* - \nu k^2 u^+ + f(k). \tag{35}$$

In this case the continuum limit is much more similar to what should be the continuum limit of a shell model describing 2D turbulence. The only two conserved quantities are both positive-definite and coincide with energy and with a generalized enstrophy Ω_α. A much more detailed investigation of both models is postponed to a forthcoming study.

7   Conclusions

In this paper we have performed a detailed investigation of a new class of helical-shell models. From the helical-Fourier decomposition of Navier-Stokes eqs. we have extracted four non-equivalent types of shell models having two inviscid quadratic invariants similar to the conserved Navier-Stokes quantities. Two of these four models coincide with the 3D and 2D versions of one of the most interesting historical shell models: the GOY model. On the other hand, the other two (model 2 and model 3 in the text) show different and peculiar properties. Most of the numerical results presented in this paper concern model 3. This model revealed to be much more stable under changes of its free parameters than the old GOY model. Why from different (severe) truncations of Navier-Stokes eqs. one ends up with so different dynamical behaviors is certainly the most stimulating question arising from our study.

As for model 2, our preliminary numerical simulations suggest the presence of a relevant backward energy transfer leading to possibly strong deviations from the Kolmogorov scaling. A detailed study of model 2 will be reported later ([25]).

The crucial role played by inviscid invariants seems to be confirmed, specially when the phenomenological analysis suggests the presence of two simultaneous transferred quantities.

Let us notice that usual arguments based on the analysis of absolute equilibrium behavior [26, 21] in order to extrapolate strongly dissipative effects like intermittency seems to fail in this class of shell models. Indeed, starting from the analysis of absolute equilibrium one should conclude that by changing α, and therefore the dimension of the second invariant, also the inertial properties

in the dissipative case should change. This is definitely true in the GOY model but definitely false in model 3.

We have also written down the PDEs describing the semi-universal (equal for models 1,2 and 3) continuum limit. Investigations of this PDE are in progress. Let us conclude with some speculation. Such a rich behavior of these four models naturally suggests that could be important to ask the same question in the original Navier-Stokes flow: what are the characteristics of the four sub-models obtained in the full Navier-Stokes decomposition taking into account separately the four different helicity classes? Can one recognize also in the case of full 3D dynamics sub-models with important back transfer of energy (like shell model 4 and shell model 2), or sub-models with different sensibility to the presence of helicity (like shell model 1 and shell model 3)?

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FIGURE CAPTIONS

- Figure (1): Energy exchange in the one-triad system for the four models. Dashed (solid) arrows point towards the mode that receives less (more) energy.

- Figure (2): Variations with α of the energy rates in the one-triad system for the four models. For each model are shown the $\dot{E}$ of the two modes that receive energy from the unstable one, whose energy rate is always kept equal to 1.
  a): $\dot{E}_2$ (solid line) and $\dot{E}_3$ (dashed line) vs α for model 1 ($\dot{E}_1 = 1$); b): $\dot{E}_1$ (solid line) and $\dot{E}_3$ (dashed line) vs α for model 2 ($\dot{E}_2 = 1$); c): $\dot{E}_2$ (solid line) and $\dot{E}_3$ (dashed line) vs α for model 3 ($\dot{E}_1 = 1$); d): $\dot{E}_1$ (solid line) and $\dot{E}_3$ (dashed line) vs α for model 4 ($\dot{E}_2 = 1$).

- Figure (3): Helicity exchange in the one-triad system for the four models. Dashed (solid) arrows point towards the mode that receives less (more) helicity.

- Figure (4): The logarithm $\log_2(S_p(n))$ of the structures functions of model 3 vs $\log_2(k_n)$. The parameters values are $\alpha = 1$ and $\lambda = 2$.

- Figure (5): The $\zeta(p)$s of GOY model (from [14]) and model 3. The parameters values are $\alpha = 1$ and $\lambda = 2$. Error bars for data concerning model 3 take into account both statiscal and power-law fit errors.

- Figure (6): The $\zeta(p)$s of GOY model and model 3 for different parameters sets (α, λ). λ is always kept equal to 2. The $\zeta(p)$s ($p = 1, .., 7$) for the GOY case $(1, 2)$ are taken from [14]. Notice that for model 3 all data sets collapse on the same curve for different α values. Error bars for data concerning model 3 take into account both statiscal and power-law fit errors.

- Figure (7): The $\zeta(p)$s of GOY model and model 3 for different parameters sets (α, λ). α is always kept equal to 1. The $\zeta(p)$s ($p = 1, .., 7$) for the three GOY cases are taken from [14]. Notice that for one value of λ ($\lambda = 1.5$) our numerically evaluated $\zeta(p)$s are slightly different from those found in the GOY model and in model 3 for other λ values. We are not able to conclude whether this discrepancy is important or not. Error bars take into account both statiscal and power-law fit errors.

	s_1	s_2	s_3	s_4	s_5	s_6
1	+	−	−	−	−	+
2	−	−	+	−	+	−
3	−	+	−	+	−	−
4	+	+	+	+	+	+

Table 1: Helicity indeces of equations (18) for the four models.

	b	c
1	$\dfrac{\lambda^{-\alpha}-\lambda^{\alpha}}{\lambda^{\alpha+1}+\lambda}$	$\dfrac{-\lambda^{-1}-\lambda^{-\alpha-1}}{\lambda^{\alpha+1}+\lambda}$
2	$\dfrac{\lambda^{-\alpha}+\lambda^{\alpha}}{-\lambda^{\alpha+1}+\lambda}$	$\dfrac{-\lambda^{-1}-\lambda^{-\alpha-1}}{-\lambda^{\alpha+1}+\lambda}$
3	$\dfrac{-\lambda^{-\alpha}-\lambda^{\alpha}}{\lambda^{\alpha+1}+\lambda}$	$\dfrac{-\lambda^{-1}+\lambda^{-\alpha-1}}{\lambda^{\alpha+1}+\lambda}$
4	$\dfrac{\lambda^{-\alpha}-\lambda^{\alpha}}{\lambda^{\alpha+1}-\lambda}$	$\dfrac{+\lambda^{-1}-\lambda^{-\alpha-1}}{\lambda^{\alpha+1}-\lambda}$

Table 2: Coefficients of equations (18) for the four models.

	η_2	η_3
1	−1	+1
2	−1	−1
3	+1	−1
4	+1	+1

Table 3: Factors in the equations (24) for the four models.